BEHAVIOR OF THE GALI INDICES FOR PERIODIC ORBITS

T. MANOS†, Ch. SKOKOS*, Ch. ANTONOPOULOS‡ and T. BOUNTIS♯

† Dipartimento di Energetica "Sergio Stecco", Università di Firenze, Firenze, ITALY.

*Max Planck Institute for the Physics of Complex Systems, Dresden, GERMANY.

‡ Faculté des Sciences, Service de Physique Non-Linéaire et Mécanique Statistique, Université Libre de Bruxelles, BELGIUM.

♯ Center for Research on Applied Nonlinear Systems (CRANS), Department of Mathematics, University of Patras, GREECE.

E-mails: thanosm@master.math.upatras.gr hskokos@pks.mpg.de cantonop@ulb.ac.be bountis@math.upatras.gr

ABSTRACT

We use the Generalized Alignment Index (GALI) to investigate the dynamics of periodic orbits and their neighborhood in several conservative nonlinear dynamical systems. For stable periodic orbits we show that GALIs tend to zero following particular power laws for Hamiltonian flows, while they fluctuate around non-zero values for symplectic maps. On the other hand, the GALIs of unstable periodic orbits tend exponentially to zero. The behavior of GALIs for orbits close to periodic ones is also studied. It is shown that, for chaotic orbits in the vicinity of unstable periodic orbits, which are influenced by the corresponding homoclinic tangle, the GALIs can exhibit a remarkable oscillatory behavior changing their values by many orders of magnitude. Exploiting the advantages of GALIs we produce phase space portraits, which are clearly depicting the dynamical changes in the neighborhood of periodic orbits when they undergo a stability change.

I. INTRODUCTION – DEFINITION OF THE GALI

We study the behavior of the ***Generalized ALignment Index*** (***GALI***) for different types of ***periodic orbits*** in multi-dimensional dynamical systems.

Considering an $2N$ – dimensional system, we follow the evolution of k deviation vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ with $2 \le k \le 2N$. Then, the *GALI* of order k, is [1]:

$$GALI_k(t) = \left\| \hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \ldots \wedge \hat{v}_k(t) \right\| \quad (1)$$

and corresponds to the volume of the generalized parallelepiped, whose edges are the k unit deviation vectors vectors (the hat ^ denotes that a vector is of unit length) and t is the continuous or discrete time.

Behavior of the *GALI* for chaotic motion and unstable periodic orbits (UPOs):

$\boldsymbol{GALI_k}$ tends exponentially to zero with exponents that involve the values of the first k largest ***Lyapunov Exponents*** $\sigma_1, \sigma_2, \ldots, \sigma_k$:

$$GALI_k(t) \sim e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)]t} \quad (2)$$

Behavior of the *GALI* for regular motion on an s-dimensional torus:

$\boldsymbol{GALI_k}$ remains essentially ***constant when*** $2 \le k \le s$, while it ***tends to zero for*** $s < k \le 2N$ ***following a power law*** [2,3]:

$$GALI_k(t) \sim \begin{cases} \text{constant}, & \text{if } 2 \le k \le s \\ \dfrac{1}{t^{k-s}}, & \text{if } s<k \le 2N-s \\ \dfrac{1}{t^{2(k-N)}}, & \text{if } 2N-s < k \le 2N \end{cases} \quad (3)$$

Behavior of the *GALI* for stable periodic orbits (SPOs):

$$GALI_k(t) \sim \begin{cases} \dfrac{1}{t^{k-1}}, & \text{if } 2 \le k < 2N \\ \dfrac{1}{t^{2N}}, & \text{if } k = 2N \end{cases} \quad (4)$$

II. THE MODELS:

a. 2 Degree of Freedom (DOF) Hénon-Heiles system

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3$$

b. 3 Degree of Freedom Hamiltonian system

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(x^2 + y^2 + z^2) - \varepsilon x z^2 - \eta y z^2$$

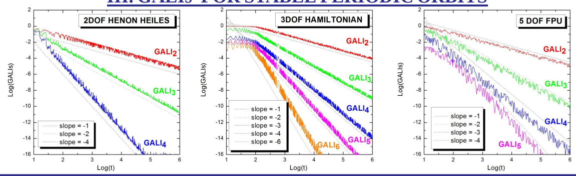
c. N=5 Degree of Freedom Hamiltonian (Fermi-Pasta-Ulam model)

$$H = \frac{1}{2}\sum_{j=1}^{N} \dot{x}_j^2 + \sum_{j=0}^{N} \left\{ \frac{1}{2}(x_{j+1} - x_j)^2 + \frac{1}{4}(x_{j+1} - x_j)^4 \right\}$$

d. 2D Hénon Map

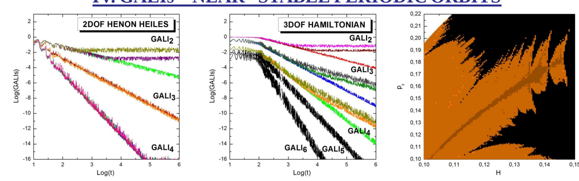
$$x' = x\cos(2\pi\omega) + (y + x^2)\sin(2\pi\omega)$$
$$y' = -x\sin(2\pi\omega) + (y + x^2)\sin(2\pi\omega)$$

III. GALIs FOR STABLE PERIODIC ORBITS



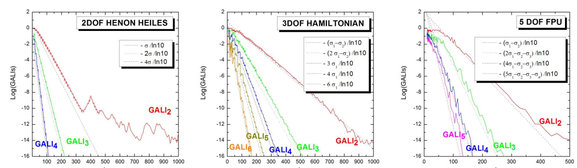
For Hamiltonian flows **all** *GALIs* decay **linearly** to zero following the laws Described in Eq. (4).

IV. GALIs "NEAR" STABLE PERIODIC ORBITS



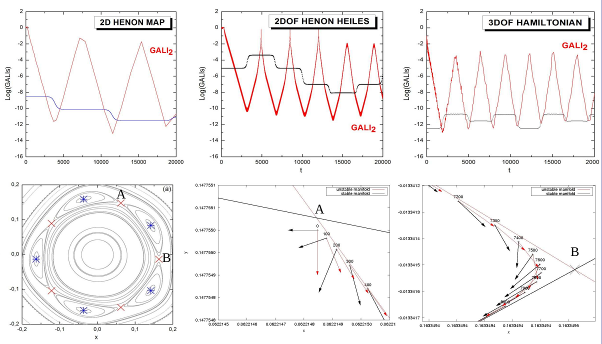
Different *GALIs*' decay as the stable periodic orbit is perturbed and gradually becoming a regular orbit lying on a torus that surrounds the SPO. The right panel shows the final values of the $GALI_2$ for a grid of initial conditions on the plane (H,p_y) for the Hénon-Heiles model, around a stable periodic orbit as the energy H varies. The brown color corresponds to the SPO while the orange to regular orbits having larger $GALI_2$ values than the SPO. As H increases the periodic orbit changes its stability from stable to unstable and eventually the motion becomes chaotic. The black color corresponds to this chaotic region.

V. GALIs FOR UNTABLE PERIODIC ORBITS



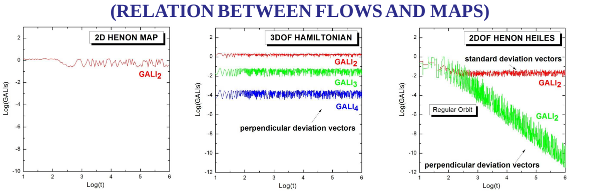
All *GALIs* decay exponentially following Eq. (2). In the left panel $GALI_2$ changes its exponential decay after some transient time (t ~500), when the orbit enters a chaotic region of different Lyapunov Exponent values.

VI. NEIGHBORHOOD OF UNSTABLE PERIODIC ORBITS



Large fluctuations of $GALI_2$ for orbits that are very close to UPOs. Upper panels: The $GALI_2$ values decrease when the orbit moves away from the UPO (blue curves show the evolution of an orbits' coordinate in arbitrary units) and increase when it approaches the UPO. This behavior is clearly seen in the case of the 2D Hénon Map where we can easily plot the evolution of the deviation vectors. In the lower left panel we show the location of an initial condition very close to an UPO of multiplicity 5 (red points). This orbit follows the stable and the unstable manifold of the UPO, as it is seen in the rest lower panels where we also present the evolution of the deviation vectors close to points A and B.

VII. GALIs FOR DEVIATION VECTORS PERPENDICULAR TO THE HAMILTONIAN FLOW (RELATION BETWEEN FLOWS AND MAPS)



Left panel: $GALI_2$ → **CONSTANT (not linear decay)** for an SPO of the *2D* Hénon Map. Using the deviation vectors perpendicular to the Hamiltonian flow instead of the standard ones the *GALIs*' behavior coincide with their behavior for maps, e.g. for periodic orbits all *GALIs* remain constant in time instead of tending to zero following a power law.

References

[1] Skokos Ch. et al., Physica D, 231, 30, 2007.

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[3] Bountis T. et al, , Journal of Computational and Applied Mathematics, 227, 17-26, 2009.